Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Analytical Description of Aircraft Motion over the Earth

Donald L. Hitzl* and John A. Breakwell†
Lockheed Research Laboratory, Palo Alto, Calif.

THE ground track motion for an aircraft flying at constant altitude and velocity in a steady turn is analyzed. First, the coupled nonlinear differential equations of motion in latitude, longitude, and heading are derived. Since analytic integration of this set is seen to be rather difficult, a solution is then constructed geometrically by use of spherical trigonometry.‡ This solution is shown to satisfy the differential equations.

Next, a second *independent* solution is constructed via vector analysis. This solution is shown in the original full-length paper (available from the authors) to be equivalent to the first solution since, by operating only with known identities, the spherical triangle solution can be transformed to the vector solution. Hence, both solutions verify the equations of motion and, moreover, are identical with each other. Finally, two interesting special cases – motion in straight and level flight and motion with constant compass heading – are mentioned; these are discussed in detail in the full paper.

Equations of Motion

As shown in Fig. 1, an earth-centered rotating coordinate system is defined with $\hat{x}$ (unit vector) along the Greenwich meridian, $\hat{z}$ along the polar axis, and $\hat{y}$ to complete a right-handed set. Also, an aircraft-centered system is defined with $\hat{r}$ along the radius vector, $\hat{e}$ to the east, and $\hat{n}$ pointing to the north. Note that the heading ψ is measured positive clockwise from north. Letting ω denote the angular velocity of the aircraft frame relative to the Earth frame, the aircraft velocity v is

$$v = \omega \times r \qquad (1)$$

Hence, noting that $(r \cdot \omega) < 0$ in Fig. 1, we get

$$r \times v = r^2 \omega - (r \cdot \omega) r = r^2 (\omega + \omega_r r) \qquad (2)$$

so

$$\omega = \frac{1}{r} [\hat{r} \times v] - \omega_r \hat{r} = -\omega_r \hat{r} - \frac{v}{r} c_\psi \hat{e} + \frac{v}{r} s_\psi \hat{n} \qquad (3)$$

Here, $\omega_r\ (>0)$ is the instantaneous rate of turn about $-\hat{r}$ and $s_\psi \triangleq \sin\psi$; $c_\psi \triangleq \cos\psi$.

Now, to obtain the desired differential equations, we write

$$\omega(\dot\psi, \dot L, \dot\lambda) = \begin{bmatrix} -\dot\psi \\ 0 \\ 0 \end{bmatrix} + M_y(-L) \begin{bmatrix} 0 \\ -\dot L \\ 0 \end{bmatrix} + M_y(-L) M_z(\lambda) \begin{bmatrix} 0 \\ 0 \\ \dot\lambda \end{bmatrix}$$

$$= (-\dot\psi + s_L \dot\lambda)\hat{r} - \dot L \hat{e} + c_L \dot\lambda \hat{n} \qquad (4)$$

where $M_x(\theta)$ denotes the 3×3 rotation matrix through the angle θ about the instantaneous axis x. Hence, on comparing Eqs. (3) and (4)

$$dL/dt = (v/r) c_\psi \qquad (5a)$$

$$d\lambda/dt = (v/r) s_\psi / c_L \qquad (5b)$$

$$d\psi/dt = (v/r) \tan L s_\psi + \omega_r \qquad (5c)$$

the desired equations of motion are obtained. Note that (a) v and r are both constant. (b) ω_r and ψ are positive in the opposite sense as r; hence the minus sign, which was incorporated in Eq. (2). (c) If $\omega_r = 0$, we have the first integral

$$c_L s_\psi = \text{constant} = c_{L_0} s_{\psi_0} \qquad (6)$$

where the subscripts *0* denote values at some initial state. Even in the special case $\omega_r = 0$, the complete analytic integration of this set is seen to be difficult. Consequently, the desired solution is constructed geometrically.

Solution from Spherical Geometry

Consider an aircraft flying at constant altitude in a steady turn about a center point c. The aircraft maintains an angular separation α from the turn axis A through c. Letting Ω denote the angular velocity vector along A, we have

$$v = \Omega r \sin\alpha \qquad (7)$$

and

$$\omega_r = \Omega \cdot \hat{r} = \Omega \cos\alpha \qquad (8)$$

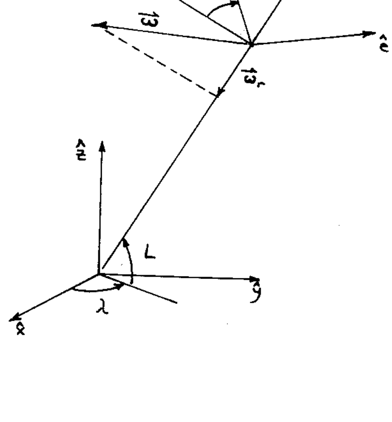

Fig. 1 Earth- and aircraft-centered coordinate systems.

Received December 16, 1974; revision received August 1, 1975.
Index categories: Navigation, Control, and Guidance Theory; Computer Technology and Computer Simulation Techniques.

*Research Scientist, Mathematics and Systems Analysis Laboratory. Member AIAA.

†Scientist, Guidance and Control Department.

‡This form of the solution is due to Prof. J. V. Breakwell of Stanford University.

which gives

$$\tan \alpha = v / r\omega_r \tag{9}$$

for the angular separation α.

The geometrical situation is shown in Fig. 2; P denotes the North Pole, 0 the initial position, and 1 a later position on the circular path. This figure then yields the following relationships:

$$\beta_0 = \psi_0 + \pi/2; \quad \beta_1 = (3\pi/2) - \psi \tag{10a}$$

$$\varphi_1 = -\varphi_0 + \Delta\varphi = -\varphi_0 + \Omega t \tag{10b}$$

$$L_c = \pi/2 - \gamma; \quad \lambda_c = \lambda_0 + \Delta\lambda_0 \tag{10c}$$

$$\lambda(t) = \lambda_0 + \Delta\lambda_0 + \Delta\lambda_1 \tag{10d}$$

The law of cosines gives

$$s_{L(t)} = c_\alpha c_\gamma + s_\alpha s_\gamma c\varphi_1 \tag{11}$$

while the law of sines yields

$$s_{\Delta\lambda_1} / s_\alpha = -c_\psi / s_\gamma = s\varphi_1 / c_L \tag{12}$$

so $L(t)$, $\lambda(t)$, and $\psi(t)$ are all determined.

Differentiating Eq. (11) yields

$$c_L \dot{L} = -s_\alpha s_\gamma s_{\varphi_1} \Omega$$

$$= -s_\alpha s_\gamma s_{\varphi_1} \frac{\omega_r}{c_\alpha} = -\frac{v}{r} s_\gamma s_{\varphi_1} = \frac{v}{r} c_L c_\psi \tag{13}$$

so $\dot{L}$ in Eq. (5a) is satisfied.

Differentiating $s_{\Delta\lambda_1} c_L = s_\alpha s_{\varphi_1}$ from Eq. (12) gives

$$c_{\Delta\lambda_1} c_L \dot{\lambda} - s_{\Delta\lambda_1} s_L \dot{L} = s_\alpha c_{\varphi_1} \Omega$$

Using Eqs. (7) and (13) and the identity

$$c_{\varphi_1} = c_{\Delta\lambda_1} s_\psi - s_{\Delta\lambda_1} c_\psi s_L \tag{14}$$

yields

$$c_L \dot{\lambda} = \frac{v}{r c_{\Delta\lambda_1}} [s_{\Delta\lambda_1} s_L c_\psi +$$

$$+ c_{\Delta\lambda_1} s_\psi - s_{\Delta\lambda_1} c_\psi s_L] = \frac{v}{r} s_\psi \tag{15}$$

so $\dot{\lambda}$ in Eq. (5b) is verified. Finally, differentiating $-s_\alpha c_\psi = s_\gamma s_{\Delta\lambda_1}$ from Eq. (12) yields

$$s_\alpha s_\psi \dot{\psi} = s_\gamma c_{\Delta\lambda_1} \dot{\lambda} \tag{16}$$

Using the identities

$$c_\alpha = c_\gamma s_L + s_\gamma c_L c_{\Delta\lambda_1} \tag{17a}$$

$$c_\gamma = c_\alpha s_L - s_\alpha c_L s_\psi \tag{17b}$$

then gives

$$s_\gamma c_{\Delta\lambda_1} = \frac{c_\alpha - c_\gamma s_L}{c_L} = c_\alpha c_L + s_\alpha s_L s_\psi$$

so

$$\dot{\psi} = [s_L + (\cot\alpha) c_L / s_\psi] \dot{\lambda} \tag{18a}$$

$$= (v/r) \tan L s_\psi + \omega_r \tag{18b}$$

again satisfying Eq. (5c).

Now, with one form of solution available, an alternative solution will be constructed using vector methods.

Vector Solution

Whereas the previous solution was obtained almost totally from Fig. 2, the following strategy was used to derive the second solution. The initial conditions (L_0, λ_0, and ψ_0—i.e., $\hat{r}_0$ and $\hat{v}_0$) were used to define the turn axis A through c and a turn radius $\boldsymbol{R}$ perpendicular to A. As shown in Fig. 3, $\boldsymbol{R} \times \boldsymbol{A}$ then yields a third vector $\boldsymbol{S}$, so the general aircraft position vector $\boldsymbol{r}(t)$ can be written as a linear combination of $\boldsymbol{A}$, $\boldsymbol{R}$, and $\boldsymbol{S}$. Transforming $\boldsymbol{r}(t)$ to earth-centered coordinates then gives $L(t)$ and $\lambda(t)$ specifying the desired motion of the ground track.

To determine $\boldsymbol{A}$, let $\boldsymbol{r}_0 + \boldsymbol{R}_0 = \boldsymbol{A}$ with $\boldsymbol{r}_0 \cdot \boldsymbol{R}_0 = 0$. The direction $\hat{R}_0$ to c is given by

$$\hat{R}_0 = \hat{v}_0 \times \hat{r}_0 = c_{\psi_0} \hat{e}_0 - s_{\psi_0} \hat{n}_0 \tag{19}$$

where $\hat{r}_0$, $\hat{e}_0$, $\hat{n}_0$ are aircraft-centered unit vectors defined from the known initial position at entry into the turn at $t = t_0 = 0$. A unit vector $\hat{A}$ along the turn axis is

$$\hat{A} = c_\alpha \hat{r}_0 + s_\alpha \hat{R}_0 \tag{20}$$

where α is the (constant) angular separation between $\boldsymbol{r}(t)$ and $\boldsymbol{A}$. Next, the turn radius $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \hat{r}_0 - c_\alpha \hat{A} \tag{21}$$

since $\boldsymbol{R} \cdot \hat{A} = 0$. Inserting Eqs. (19) and (20)

$$\boldsymbol{R} = s_\alpha [s_\alpha \hat{r}_0 - c_\alpha c_{\psi_0} \hat{e}_0 + c_\alpha s_{\psi_0} \hat{n}_0] = s_\alpha \hat{\boldsymbol{R}} \tag{22}$$

so, referring to Fig. 3

$$\hat{S} = \hat{R} \times \hat{A} = s_{\psi_0} \hat{e}_0 + c_{\psi_0} \hat{n}_0 \tag{23}$$

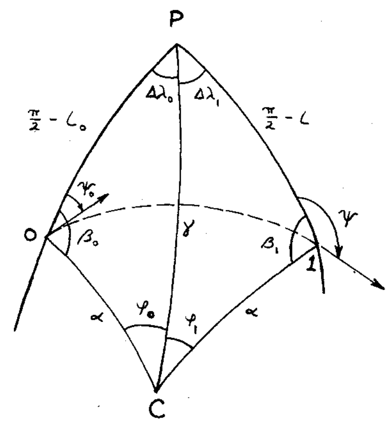

Fig. 2 Spherical triangles defining motion in a steady turn.

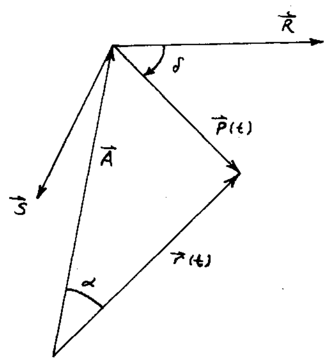

Fig. 3 Construction of the position vector $r(t)$.

and the unit position vector $\hat{r}(t)$ can be written

$$\hat{r}(t) = c_\alpha \hat{A} + s_\alpha \hat{P}(t) = c_\alpha \hat{A} + s_\alpha (c_\delta \hat{R} + s_\delta \hat{S}) \quad (24a)$$

$$\triangleq \rho \hat{r}_0 + \sigma \hat{e}_0 + \tau \hat{n}_0 \quad (24b)$$

where $\delta = \delta(t) = \Omega t = (\omega_r t)/\cos\alpha$ and

$$\rho = 1 - s_\alpha^2 (1 - c_\delta); \quad \sigma = s_\alpha [c_\alpha c_{\psi_0}(1 - c_\delta) + s_\delta s_{\psi_0}] \quad (25a,b)$$

$$\tau = -s_\alpha [c_\alpha s_{\psi_0}(1 - c_\delta) - s_\delta c_{\psi_0}] \quad (25c)$$

Note that $\hat{r}(t_0) = \hat{r}_0$.

Using the rotation matrices $M_y(-L)$ and $M_z(\lambda)$, which appeared previously in Eq. (4), $\hat{r}(t)$ is transformed to an earth-centered coordinate system

$$\hat{r}(t) = \xi \hat{x} + \eta \hat{y} + \zeta \hat{z} \quad (26)$$

where

$$\zeta = \sin L(t) = \rho s_{L_0} + \tau c_{L_0} \quad (27)$$

and

$$\frac{\eta}{\xi} = \tan \lambda(t) = \frac{\tan\lambda_0 + \sigma/(\rho c_{L_0} - \tau s_{L_0})}{1 - (\tan\lambda_0)\sigma/(\rho c_{L_0} - \tau s_{L_0})} \quad (28a)$$

$$\triangleq \tan[\lambda_0 + \Delta\lambda(t)] \quad (28b)$$

so

$$\tan \Delta\lambda = \sigma/(\rho c_{L_0} - \tau s_{L_0}) \quad (29)$$

Thus, an alternative determination of the ground track is given by Eqs. (27) and (29). As mentioned earlier, this solution is equivalent to the previous solution (which we already know satisfies the differential equations), since both forms of the solution can be transformed one into the other. This transformation is presented in detail in the full paper.

Conclusions

The sole purpose for the preceding development was the need for an analytical description of the ground track for a real aircraft flying in a coordinated turn. In fact, this ground track motion formed the foundation of a target acquisition and pointing program used during a series of flights with an instrumented research aircraft.

On the Wave Drag Integral for Slender Bodies

Rajendra K. Bera*
National Aeronautical Laboratory, Bangalore, India

Introduction

THE transonic wave drag integral derived from linear theory gives results of engineering accuracy and has therefore found wide application in preliminary design. Of the several methods now available[1-5] for evaluating the drag integral, Eminton's method enjoys some popularity because of its basic simplicity. In this Note we suggest an alternative to the Eminton method and use Filon's quadrature rule to calculate the wave drag, and compare it against the Eminton method and a collocation method.

Received May 2, 1975; revision received August 11, 1975.
Index categories: Aircraft Aerodynamics (including Component Aerodynamics); Subsonic and Transonic Flow.

*Scientist, Aerodynamics Division, and Head, Information / Coordination Division.

Calculation Procedure

The wave drag integral I, for a sufficiently slender body at zero incidence is given by[1]

$$I \equiv D/q = -(\tfrac{1}{2}\pi) \int_0^1 \int_0^1 S''(x) S''(\xi) \log|x - \xi| \, dx \, d\xi \quad (1)$$

subject to the auxiliary conditions that

$$S'(0) = S'(1) = 0 \quad (2)$$

and that the body be sufficiently smooth so that $S'(x)$ is continuous in $0 \leq x \leq 1$. In Eq. (1) D is the wave drag, q the freestream dynamic pressure, $S(x)$ is the body cross-section area distribution, and x is measured from the body nose in the direction of the base along the body axis of symmetry. The body is normalized to unit length and primes denote differentiation with respect to the function argument.

Following Eminton, $S'(x)$ is represented by a Fourier sine series

$$S'(x) = \sum_{r=1}^{\infty} a_r \sin r\theta \quad (3)$$

where

$$x = \tfrac{1}{2}(1 - \cos\theta) \quad (4)$$

which automatically satisfies the auxilliary conditions imposed on $S(x)$.

The substitution of Eq. (3) in Eq. (1) results in

$$D/q = (\pi/4) \sum_{r=1}^{\infty} r \, a_r^2 \quad (5)$$

and an integration of Eq. (3) shows

$$S(x) = a_0 + \tfrac{1}{4} a_1 \theta + \tfrac{1}{4} \sum_{r=2}^{\infty} (a_{r+1} - a_{r-1}) \sin r\theta / r \quad (6)$$

The first three coefficients a_0, a_1, and a_2 may be shown to be equal to

$$a_0 = S(0) \quad (7)$$

$$a_1 = 4[S(1) - S(0)]/\pi \quad (8)$$

$$a_2 = 8[2V - S(1) - S(0)]/\pi \quad (9)$$

where V is the body volume. For a given body the nose area $S(0)$, and the base area $S(1)$ will be known, and frequently V will also be known. The remaining or all of the a_r may be obtained from

$$a_r = (2/\pi) \int_0^\pi S'(x) \sin r\theta \, d\theta; \quad r = 1,2,\ldots \quad (10)$$

However, a practical difficulty exists since $S'(x)$ is seldom known in an analytical form or with sufficient accuracy in numerical form at a reasonably large number of points.

To overcome this difficulty, we multiply Eq. (6) by $\sin k\theta$ and integrate over 0 to π to obtain

$$\int_0^\pi S(\theta) \sin k\theta \, d\theta + (\cos k\pi - 1) S(0)/k + a_1 \pi \cos k\pi / 4k$$

$$= \pi(a_{k+1} - a_{k-1})/8k$$